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Spaces of real polynomials with common roots

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Let $RX^l_{k,n}$ be the space consisting of all $(n+1)$–tuples $(p_0(z),\dots,p_n(z))$ of monic polynomials over $\mathbb{R}$ of degree k and such that there are at most l roots common to all $p_i(z)$. In this paper, we prove a stable splitting of $RX^l_{k,n}$.

55P15; 55P35, 58D15

1 Introduction

Let $\mathrm{Rat}_k(\mathbb{C}P^n)$ denote the space of based holomorphic maps of degree k from the Riemannian sphere $S^2=\mathbb{C}\cup\infty$ to the complex projective space $\mathbb{C}P^n$. The basepoint condition we assume is that $f(\infty)=[1,\dots,1]$. Such holomorphic maps are given by rational functions:

$$\begin{aligned}\mathrm{Rat}_k(\mathbb{C}P^n)=\{(p_0(z),\dots,p_n(z)) : &\text{ each } p_i(z) \text{ is a monic polynomial over } \mathbb{C}\\ &\text{of degree } k \text{ and such that there are no roots common to all } p_i(z)\}.\end{aligned}$$

There is an inclusion $\mathrm{Rat}_k(\mathbb{C}P^n)\hookrightarrow \Omega^2_k\mathbb{C}P^n\simeq\Omega^2S^{2n+1}$. Segal [6] proved that the inclusion is a homotopy equivalence up to dimension $k(2n-1)$. Later, the stable homotopy type of $\mathrm{Rat}_k(\mathbb{C}P^n)$ was described by Cohen et al [2, 3] as follows. Let $\Omega^2S^{2n+1}\underset{s}{\simeq}\bigvee_{1\le q}D_q(S^{2n-1})$ be Snaith's stable splitting of Ω^2S^{2n+1}. Then

$$\mathrm{Rat}_k(\mathbb{C}P^n)\underset{s}{\simeq}\bigvee_{q=1}^{k}D_q(S^{2n-1}). \tag{1–1}$$

In Kamiyama [4], (1–1) was generalized as follows. We set

$$\begin{aligned}X^l_{k,n}=\{(p_0(z),\dots,p_n(z)) : &\text{ each } p_i(z) \text{ is a monic polynomial over } \mathbb{C}\\ &\text{of degree } k \text{ and such that there are at most } l \text{ roots common to all } p_i(z)\}.\end{aligned}$$

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In particular, $X^0_{k,n} = \mathrm{Rat}_k(\mathbb{C}P^n)$. Let

$$J^l(S^{2n}) \simeq S^{2n} \cup e^{4n} \cup e^{6n} \cup \ldots \cup e^{2ln} \subset \Omega S^{2n+1}$$

be the lth stage of the James filtration of ΩS^{2n+1}, and let $W^l(S^{2n})$ be the homotopy theoretic fiber of the inclusion $J^l(S^{2n}) \hookrightarrow \Omega S^{2n+1}$. We generalize Snaith's stable splitting of $\Omega^2 S^{2n+1}$ as follows:

$$W^l(S^{2n}) \underset{s}{\simeq} \bigvee_{1 \leq q} D_q \xi^l(S^{2n}).$$

Then we have a stable splitting

$$X^l_{k,n} \underset{s}{\simeq} \bigvee_{q=1}^{k} D_q \xi^l(S^{2n}).$$

The purpose of this paper is to study the real part $RX^l_{k,n}$ of $X^l_{k,n}$ and prove a stable splitting of this. More precisely, let $RX^l_{k,n}$ be the subspace of $X^l_{k,n}$ consisting of elements $(p_0(z), \ldots, p_n(z))$ such that each $p_i(z)$ has real coefficients. Our main results will be stated in Section 2. Here we give a theorem which generalizes (1–1). Since the homotopy type of $RX^0_{k,1}$ is known (see Example 2.1 (iii)), we assume $n \geq 2$. In this case, there is an inclusion

$$RX^0_{k,n} \hookrightarrow \Omega S^n \times \Omega^2 S^{2n+1}.$$

(See Lemma 3.1.)

Theorem 1.1 *For $n \geq 2$, we define the weight of stable summands in ΩS^n as usual, but those in $\Omega^2 S^{2n+1}$ we define as being twice the usual one. Then $RX^0_{k,n}$ is stably homotopy equivalent to the collection of stable summands in $\Omega S^n \times \Omega^2 S^{2n+1}$ of weight $\leq k$. Hence,*

$$RX^0_{k,n} \underset{s}{\simeq} \bigvee_{p+2q \leq k} \Sigma^{p(n-1)} D_q(S^{2n-1}) \vee \bigvee_{p=1}^{k} S^{p(n-1)}.$$

This paper is organized as follows. In Section 2 we state the main results. We give a stable splitting of $RX^l_{k,n}$ in Theorem A and Theorem B. In order to prove these theorems, we also consider a space $Y^l_{k,n}$, which is an open set of $RX^l_{k,n}$. We give a stable splitting of $Y^l_{k,n}$ in Proposition C. In Section 3 we prove Proposition C. In Section 4 we prove Theorem A and Theorem B.

2 Main results

We set

$$Y^l_{k,n} = \{(p_0(z), \dots, p_n(z)) \in RX^l_{k,n} : \text{there are no } \textit{real} \text{ roots common to all } p_i(z)\}.$$

The spaces $Y^l_{k,n}$ and $RX^l_{k,n}$ are in the following relation:

$$\begin{array}{ccccccccccc}
Y^k_{k,n} & \supset & Y^{k-1}_{k,n} & \supset & \cdots & \supset & Y^l_{k,n} & \supset & \cdots & \supset & Y^1_{k,n} & = & Y^0_{k,n} \\
\cap & & \cap & & & & \cap & & & & \cap & & \| \\
RX^k_{k,n} & \supset & RX^{k-1}_{k,n} & \supset & \cdots & \supset & RX^l_{k,n} & \supset & \cdots & \supset & RX^1_{k,n} & \supset & RX^0_{k,n}
\end{array}$$

where each subset is an open set. Moreover, $Y^{2i+1}_{k,n} = Y^{2i}_{k,n}$. In fact, if $\alpha \in H_+$ (where H_+ is the open upper half-plane) is a root of a real polynomial, then so is $\overline{\alpha} \in H_-$.

We have the following examples.

Example 2.1

(i) It is proved by Mostovoy [5] that $Y^k_{k,1}$ consists of $k+1$ contractible connected components.

(ii) The following result is proved by Vassiliev [7]. For $n \geq 3$, there is a homotopy equivalence $Y^k_{k,n} \simeq J^k(S^{n-1})$, where $J^k(S^{n-1})$ is as above the kth stage of the James filtration of ΩS^n. For $n = 2$, these spaces are stably homotopy equivalent.

(iii) It is proved by Segal [6] that

$$RX^0_{k,1} \simeq \coprod_{q=0}^{k} \mathrm{Rat}_{\min(q,k-q)}(\mathbb{C}P^1).$$

(iv) $RX^{k-1}_{k,n} \cong \mathbb{R}^k \times (\mathbb{R}^{kn})^*$ and $RX^k_{k,n} \cong \mathbb{R}^{k(n+1)}$.

In fact, $(p_0(z), \dots, p_n(z)) \in RX^k_{k,n}$ is an element of $RX^{k-1}_{k,n}$ if and only if $p_i(z) \neq p_j(z)$ for some i, j. Hence, the first homeomorphism holds.

Now we state our main results.

Theorem A *For $n \geq 1$ and $i \geq 0$, there is a homotopy equivalence*

$$RX^{2i+1}_{k,n} \simeq X^i_{[\frac{k}{2}],n},$$

where $\left[\frac{k}{2}\right]$ denotes as usual the largest integer $\leq \frac{k}{2}$.

Theorem B *For $n \geq 1$ and $i \geq 0$, there is a stable homotopy equivalence*

$$RX^{2i}_{k,n} \underset{s}{\simeq} X^i_{[\frac{k}{2}],n} \vee \Sigma^{2in}\Big(\bigvee_{\substack{p+2q\leq k-2i \\ 1\leq p}} \Sigma^{p(n-1)} D_q(S^{2n-1}) \vee \bigvee_{p=1}^{k-2i} S^{p(n-1)} \Big).$$

We study $RX^l_{k,n}$ by induction with making l larger. Hence, the induction starts from $RX^0_{k,n}$. Recall that $RX^0_{k,n} = Y^0_{k,n}$. We study $Y^l_{k,n}$ by induction with making l smaller, where the initial condition is given in Example 2.1 (ii). In fact, we have the following proposition.

Proposition C

(i) *For $n \geq 2$, we define the weight of stable summands in ΩS^n as usual, but those in $W^i(S^{2n})$ we define as being twice the usual one. Then $Y^{2i}_{k,n}$ is stably homotopy equivalent to the collection of stable summands in $\Omega S^n \times W^i(S^{2n})$ of weight $\leq k$. Hence,*

$$Y^{2i}_{k,n} \underset{s}{\simeq} \bigvee_{p+2q\leq k} \Sigma^{p(n-1)} D_q \xi^i(S^{2n}) \vee \bigvee_{p=1}^{k} S^{p(n-1)}.$$

(ii) *When $n = 1$, there is a homotopy equivalence*

$$Y^{2i}_{k,1} \simeq \coprod_{q=0}^{k} X^i_{\min(q,k-q),1}.$$

Note that Proposition C (ii) is a generalization of Example 2.1 (i) and (iii).

3 Proof of Proposition C

We study the space of continuous maps which contains $Y^k_{k,n}$ or $RX^0_{k,n}$. For simplicity, we assume $n \geq 2$. (The case for $n = 1$ can be obtained by slight modifications.) Each $f \in Y^k_{k,n}$ defines a map $f\colon S^1 \to \mathbb{R}P^n$, where $S^1 = \mathbb{R} \cup \infty$. Hence, there is a natural map

$$Y^k_{k,n} \longrightarrow \Omega_{k \bmod 2} \mathbb{R}P^n \simeq \Omega S^n.$$

Example 2.1 (ii) implies that $Y^k_{k,n}$ is the $k(n-1)$–skeleton of ΩS^n.

On the other hand, let $\mathrm{Map}_k^T(\mathbb{C}P^1, \mathbb{C}P^n)$ be the space of continuous basepoint-preserving conjugation-equivariant maps of degree k from $\mathbb{C}P^1$ to $\mathbb{C}P^n$. Then there is an inclusion

$$RX^0_{k,n} \hookrightarrow \mathrm{Map}_k^T(\mathbb{C}P^1, \mathbb{C}P^n).$$

Lemma 3.1 *For $n \geq 2$, $\mathrm{Map}_k^T(\mathbb{C}P^1, \mathbb{C}P^n) \simeq \Omega S^n \times \Omega^2 S^{2n+1}$.*

Proof It is easy to see that

$$\mathrm{Map}_k^T(\mathbb{C}P^1, \mathbb{C}P^n) \simeq \mathrm{Map}_0^T(\mathbb{C}P^1, \mathbb{C}P^n).$$

Since $\mathrm{Map}_0^T(\mathbb{C}P^1, \mathbb{C}P^n)$ can be thought as the space of maps

$$(D^2, S^1, *) \to (\mathbb{C}P^n, \mathbb{R}P^n, *)$$

of degree 0, there is a fibration

$$\Omega^2 S^{2n+1} \to \mathrm{Map}_0^T(\mathbb{C}P^1, \mathbb{C}P^n) \to \Omega S^n.$$

This is a pullback of the path fibration $\Omega^2 S^{2n+1} \to P\Omega S^{2n+1} \to \Omega S^{2n+1}$ by the map $\Omega f\colon \Omega S^n \to \Omega S^{2n+1}$, where $f\colon S^n \to S^{2n+1}$ is a lift of the inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$. Since f is null homotopic, the fibration is trivial. This completes the proof of Lemma 3.1. □

Hereafter, every homology is with $\mathbb{Z}/p$–coefficients, where p is a prime. Recall that for $n \geq 2$, we have $H_*(\Omega S^n) \cong \mathbb{Z}/p\left[x_{n-1}\right]$. We define the weight of x_{n-1} by $w(x_{n-1}) = 1$. On the other hand, we define the weight of an element of $H_*\left(X^i_{k,n}\right)$ as being twice the usual one. For example, let $y_{2(l+1)n-1}$ be the generator of $\widetilde{H}_*\left(X^l_{k,n}\right)$ of least degree. The usual weight of $y_{2(l+1)n-1}$ is $l+1$, but we reset $w(y_{2(l+1)n-1}) = 2(l+1)$.

Proposition 3.2 *For $n \geq 2$, $H_*\left(Y^{2i}_{k,n}\right)$ is isomorphic to the subspace of $H_*\left(\Omega S^n \times X^i_{k,n}\right)$ spanned by monomials of weight $\leq k$.*

We prove the proposition from the following lemma.

Lemma 3.3 *We have the following long exact sequence:*

$$\cdots \to H_*(Y^{2i-2}_{k,n}) \to H_*(Y^{2i}_{k,n}) \xrightarrow{\phi} H_{*-2in}(RX^0_{k-2i,n}) \to H_{*-1}(Y^{2i-2}_{k,n}) \to \cdots .$$

Proof In [4, Propositions 4.5 and 5.4], we constructed a similar long exact sequence from the fact that

$$X^l_{k,n} - X^{l-1}_{k,n} = \mathbb{C}^l \times \mathrm{Rat}_{k-l}(\mathbb{C}P^n),$$

where $\mathbb{C}^l \times \mathrm{Rat}_{k-l}(\mathbb{C}P^n)$ corresponds to the subspace of $X^l_{k,n}$ consisting of elements $(p_0(z), \ldots, p_n(z))$ such that there are exactly l roots common to all $p_i(z)$. The proposition is proved similarly using the fact that

$$Y^{2i}_{k,n} - Y^{2i-2}_{k,n} \cong \mathrm{SP}^i(H_+) \times RX^0_{k-2i,n},$$

where $\mathrm{SP}^i(H_+)$ denotes the ith symmetric product of H_+. □

Proof of Theorem 3.2 In order to prove Theorem 3.2 by induction, we introduce the following total order $\leq$ to $Y^{2i}_{k,n}$ for $k \geq 1$ and $i \geq 0$: $Y^{2i}_{k,n} < Y^{2i'}_{k',n}$ if and only if

(i) $k < k'$, or

(ii) $k = k'$ and $i > i'$.

By Example 2.1 (ii), Theorem 3.2 holds for $Y^k_{k,n}$. Assuming that Theorem 3.2 holds for $Y^{2i}_{k,n}$ and $RX^0_{k-2i,n}$, we prove for $Y^{2i-2}_{k,n}$. We have the following long exact sequence:

$$\begin{aligned}(3\text{–}1)\qquad \cdots \longrightarrow H_*\Big(X^{i-1}_{[\frac{k}{2}],n}\Big) \longrightarrow H_*\Big(X^{i}_{[\frac{k}{2}],n}\Big)& \\ \xrightarrow{\ \psi\ } H_{*-2in}\Big(\mathrm{Rat}_{[\frac{k}{2}]-i}(\mathbb{C}P^n)\Big) \xrightarrow{\ \theta\ } H_{*-1}\Big(X^{i-1}_{[\frac{k}{2}],n}\Big) \longrightarrow \cdots .&\end{aligned}$$

For $n \geq 2$, we consider the homomorphism

$$1 \otimes \psi\colon\ H_*(\Omega S^n) \otimes H_*\Big(X^{i}_{[\frac{k}{2}],n}\Big) \to H_*(\Omega S^n) \otimes H_{*-2in}\Big(\mathrm{Rat}_{[\frac{k}{2}]-i}(\mathbb{C}P^n)\Big).$$

Restricting the domain to $H_*(Y^{2i}_{k,n})$, we obtain the homomorphism ϕ in Lemma 3.3. Now it is easy to prove Theorem 3.2. □

Proof of Proposition C (i) We construct a stable map from the right-hand side of Proposition C (i) to $Y^{2i}_{k,n}$. Since our constructions are similar, we construct a stable map $g_{p,q,i,n}\colon\ \Sigma^{p(n-1)} D_q \xi^i(S^{2n}) \to Y^{2i}_{k,n}$. First, using the fact that $RX^0_{1,n} \simeq S^{n-1}$ (see Example 2.1 (iv)), there is a stable map $f_{p,n}\colon\ S^{p(n-1)} \to RX^0_{p,n}$. Second, there is a stable section $e_{q,i,n}\colon\ D_q \xi^i(S^{2n}) \to X^i_{q,n}$. Third, there is an inclusion

$$\eta_{q,i,n}\colon\ X^i_{q,n} \hookrightarrow Y^{2i}_{2q,n}. \qquad (3\text{–}2)$$

To construct this, we fix a homeomorphism $h : \mathbb{C} \xrightarrow{\cong} H_+$. For $(p_0(z), \dots, p_n(z)) \in X^i_{q,n}$, we write $p_j(z) = \prod_{s=1}^{q}(z - \alpha_{s,j})$. Then we set

$$\eta_{q,i,n}(p_0(z), \dots, p_n(z)) = \left(\prod_{s=1}^{q}(z - h(\alpha_{s,0}))(z - \overline{h(\alpha_{s,0})}), \dots, \prod_{s=1}^{q}(z - h(\alpha_{s,n}))(z - \overline{h(\alpha_{s,n})})\right).$$

Now consider the following composite of maps

$$S^{p(n-1)} \times D_q\xi^i(S^{2n}) \xrightarrow{f_{p,n} \times (\eta_{q,i,n} \circ e_{q,i,n})} RX^0_{p,n} \times Y^{2i}_{2q,n} \xrightarrow{\mu} Y^{2i}_{p+2q,n} \hookrightarrow Y^{2i}_{k,n}, \tag{3–3}$$

where μ is a loop sum which is constructed in the same way as in the loop sum $\mathrm{Rat}_k(\mathbb{C}P^n) \times \mathrm{Rat}_l(\mathbb{C}P^n) \to \mathrm{Rat}_{k+l}(\mathbb{C}P^n)$ in Boyer–Mann [1]. We can construct $g_{p,q,i,n}$ from (3–3).

Note that the stable map for Proposition C (i) is compatible with the homology splitting by weights. Using Theorem 3.2, it is easy to show that this map induces an isomorphism in homology, hence is a stable homotopy equivalence. This completes the proof of Proposition C (i). □

Proof of Proposition C (ii) By a similar argument to the proof of Theorem 3.2, we can calculate $H_*(Y^{2i}_{k,1})$. Then we can construct an unstable map from the right-hand side of Proposition C (ii) to $Y^{2i}_{k,1}$ in the same way as in Proposition C (i). □

4 Proof of Theorem A and Theorem B

Proposition 4.1 *The homologies of the both sides of Theorem A or Theorem B are isomorphic.*

Proof We prove the proposition about $RX^l_{k,n}$ by induction with making l larger. As in Lemma 3.3, there is a long exact sequence

$$\cdots \longrightarrow H_*\big(RX^l_{k,n}\big) \longrightarrow H_*\big(RX^{l+1}_{k,n}\big) \longrightarrow H_{*-(l+1)n}\big(RX^0_{k-(l+1),n}\big) \xrightarrow{\Theta} H_{*-1}\big(RX^l_{k,n}\big) \longrightarrow \cdots.$$

This sequence is constructed from the following decomposition as sets

$$RX_{k,n}^{l+1} - RX_{k,n}^{l} = \coprod_{a+2b=l+1} \mathrm{SP}^a(\mathbb{R}) \times \mathrm{SP}^b(H_+) \times RX_{k-(l+1),n}^{0}$$

and the fact that $H_c^*(\mathrm{SP}^a(\mathbb{R})) = 0$ for $a \geq 2$, where H_c^* is the cohomology with compact supports.

Assuming that the proposition holds for $l \leq 2i+1$, we determine $H_*\big(RX_{k,n}^{2i+2}\big)$. The homomorphism Θ is given as follows. Note that Theorem B is equivalent to

$$RX_{k,n}^{2i} \underset{s}{\simeq} X_{[\frac{k}{2}],n}^{i} \vee \Sigma^{(2i+1)n-1}\big(RX_{k-2i-1,n}^{0} \vee S^0\big). \tag{4–1}$$

From inductive hypothesis, we have

$$\begin{aligned} &H_{*-(2i+2)n}\big(RX_{k-2i-2,n}^{0}\big) \cong \\ &\qquad H_{*-(2i+2)n}\Big(\mathrm{Rat}_{[\frac{k}{2}]-(i+1)}(\mathbb{C}P^n)\Big) \oplus \widetilde{H}_{*-(2i+2)n}\big(\Sigma^{n-1}RX_{k-2i-3,n}^{0} \vee S^{n-1}\big) \end{aligned} \tag{4–2}$$

and

$$H_{*-1}(RX_{k,n}^{2i+1}) \cong H_{*-1}\Big(X_{[\frac{k}{2}],n}^{i}\Big). \tag{4–3}$$

Recall the homomorphism θ in (3–1) with i replaced by $i+1$. Then $\Theta\colon$ (4–2) $\to$ (4–3) is given by mapping the first summand by θ and the second summand by 0. Hence, $H_*\big(RX_{k,n}^{2i+2}\big)$ is isomorphic to the homology of the right-hand side of (4–1) with i replaced by $i+1$.

By a similar argument, we can determine $H_*\big(RX_{k,n}^{2i+1}\big)$ inductively by assuming the truth of the proposition for $l \leq 2i$. This completes the proof of Theorem 4.1. □

Finally, we construct an unstable map (resp. a stable map) from the right-hand side of Theorem A (resp. (4–1)) to $RX_{k,n}^{2i+1}$ (resp. $RX_{k,n}^{2i}$). First, the unstable map from the right-hand side of Theorem A or the first stable summand in (4–1) is essentially the inclusion

$$X_{q,n}^{i} \xrightarrow{\eta_{q,i,n}} Y_{2q,n}^{2i} \subset RX_{2q,n}^{2i},$$

where $\eta_{q,i,n}$ is defined in (3–2). Next, the stable map from the second stable summand in (4–1) is constructed in the same way as in $g_{p,q,i,n}$ (see (3–3)) using the fact that $RX_{2i+1,n}^{2i} \simeq S^{(2i+1)n-1}$ (see Example 2.1 (iv)). This completes the proofs of Theorem A and Theorem B. □

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